Continued fraction representation of temporal multiscaling in turbulence

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It was shown recently that the anomalous scaling of simultaneous correlation functions in turbulence is intimately related to the breaking of temporal scale invariance, which is equivalent to the appearance of infinitely many times scales in the time dependence of time-correlation functions. In this paper we derive a continued fraction representation of turbulent time correlation functions which is exact and in which the multiplicity of time scales is explicit. We demonstrate that this form yields precisely the same scaling laws for time derivatives and time integrals as the ''multi-fractal'' representation that was used before. Truncating the continued fraction representation yields the ''best'' estimates of time correlation functions if the given information is limited to the scaling exponents of the simultaneous correlation functions up to a certain, finite order. It is worth noting that the derivation of a continued fraction representation obtained here for a time evolution operator which is not Hermitian or anti-Hermitian may be of independent interest. [S1063-651X(99)05511-7]

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I. INTRODUCTION

It is commonly argued [1,2] that fully developed hydrodynamic turbulence exhibits simultaneous statistical objects whose scaling properties are anomalous. For example the so called structure functions satisfy scaling laws of the form

$$S_n(R)=\langle|U(r+R,t)-u(r,t)|^n\rangle\sim R^{\zeta n}, \qquad (1)$$

where $\langle\rangle$ stands for a suitably defined ensemble average. Here $u(r,t)$ is the Eulerian velocity field, and the ζ_n are scaling exponents which are nonlinear functions of n. The separation distance R lies in the ''inertial range,'' i.e., $\eta \ll R \ll L$ with η the inner viscous scale and L the outer integral scale of turbulence. The nonlinear dependence is referred to as ''anomalous scaling'' or ''multiscaling,'' and the issue of evaluating these exponents from either phenomenological models or from first principles has attracted significant amount of effort in the last decade.

It has only recently been discovered [3] that also the time dependence of the nth order correlation functions is multiscaling, and that ''dynamical scaling'' is broken. This phenomenon seems to distinguish turbulence from other problems in which scaling is anomalous, such as critical phenomena. In the latter case dynamical scaling is invoked by stating that a space time correlation function $F(R,t)$ is a homogeneous function of its arguments in the sense that $F(\lambda R,\lambda^z t)=\lambda^\zeta F(R,t)$, where ζ and z are the ''static'' and ''dynamic'' scaling exponents, respectively. In turbulence such relations do not exist even when the same-time correlation functions are homogeneous functions of the spatial coordinates. The importance of this fact in determining the structure of the theory has been stressed in Ref. [4], see also [5].

In this paper we address temporal multiscaling on the basis of the continued fraction representation of turbulent correlation function [6,7]. This approach will lead us to a different point of view of temporal multi-scaling, in agreement with the conclusions in Ref. [3]. The advantage of the continued fraction expansion is threefold: first, it is derived on the basis of an exact formulation of the time correlation functions and their dynamics. The phenomenon of temporal multiscaling is related in this approach to the scaling properties of higher order moments or, equivalently, of higher order temporal derivatives of the correlation functions, computed at zero time difference. Second, this approach furnishes information not only about the leading temporal scaling exponents, but also about the subleading ones. Third, a finite truncation of the continued fraction representation is in a sense the ''best'' possible representation when the information about the scaling of the set of one-time correlation functions of all moment orders n is limited to the low order scaling exponents. We will show that the scaling laws exhibited by the continued fraction representation are identical to those derivable by the ''multifractal'' representation of the time-correlation function [4], adding weight and justification to the latter. Since the multifractal representation was used recently to estimate the scaling exponents from first principles [5], we ascribe some weight to being able to justify it further.

To keep the formalism minimal and the result clearest, we treat in this paper only the second order space-time correlation function of turbulent fields. The formalism can be used to generate representations of any higher order correlation function, but we do not elaborate on this in the present text. In Sec. II we review briefly the Zwanzig-Mori formalism [8,9] which was applied to time correlation functions in turbulence [6,7], and display the continued fraction representation of the second order time correlation function of the velocity field differences. We show that the coefficients in this representation can be written in terms of time derivatives at equal times of the same second order correlation function or, equivalently, in terms of the whole set of the nth order equal time correlation functions for all n. In Sec. III we set up a theory for the evaluation of the scaling exponents of these continued fraction coefficients. In Sec. IV we derive the scaling laws implied by the continued fraction representation for the time derivatives of the correlation function evaluated at equal times. In Sec. V we relate our results to the multifractal representation of the correlations functions, and explain the scaling-equivalence of the two representations. In Sec. VI we offer a summary and a short discussion.

II. CONTINUED FRACTION REPRESENTATION

In thinking about the dynamics of turbulent flow one cannot deal with time-correlation functions of the Eulerian field since these are dominated by the kinematic sweeping time scale. We need to consider Lagrangian or Monin-Belinicher-L'vov velocity differences. We prefer the latter since they obey Navier-Stokes-like equations of motion which are local in time. In terms of the Eulerian velocity $u(r,t)$ Monin [10] and also Belinicher and L'vov [11] defined the field $v(r_0,t_0|r,t)$ as

$$v(r_0,t_0|r,t)\equiv u[r+\rho(r_0,t),t], \tag{2}$$

where

$$\rho(r_0,t)=\int_{t_0}^{t} ds\, u[r_0+\rho(r_0,s),s]. \tag{3}$$

The observation of Belinicher and L'vov [11] was that the variables $\mathcal{W}(r_0,t_0|r,r',t)$ defined as

$$\mathcal{W}(r_0,t_0|r,r',t)\equiv v(r_0,t_0|r,t)-v(r_0,t_0|r',t), \tag{4}$$

exactly satisfy a Navier-Stokes-like equation in the incompressible limit:

$$\left[\frac{\partial}{\partial t}+\vec{\vec{\mathcal{P}}}\mathcal{W}(r_0,t_0|r,r_0,t)\cdot\nabla_r+\vec{\vec{\mathcal{P}}}'\mathcal{W}(r_0,t_0|r',r_0,t)\cdot\nabla'_r \right.$$
$$\left. -\nu(\nabla_r^2+\nabla_r'^2)\right]\mathcal{W}(r_0,t_0|r,r',t)=0. \tag{5}$$

We emphasize that the application of the transversal projector $\vec{\vec{\mathcal{P}}}$ to any given vector field $a(r)$ is nonlocal, and has the form

$$[\vec{\vec{\mathcal{P}}}a(r)]^{\alpha}=\int d\tilde{r}\mathcal{P}^{\alpha\beta}(r-\tilde{r})a^{\beta}(\tilde{r}). \tag{6}$$

The explicit form of the kernel $\mathcal{P}^{\alpha\beta}(r)$ can be found, for example, in Ref. [12]. In Eq. (5) $\vec{\vec{\mathcal{P}}}$ and $\vec{\vec{\mathcal{P}}}'$ are projection operators which act on fields that depend on the corresponding coordinates r and r'. For our purpose below we introduce the Liouville operator $\hat{\mathcal{L}}(r_0,t_0|r,r')$ for the time evolution which is defined via the total time derivative of $\mathcal{W}(r_0,t_0|r,r',t)$ at time $t=t_0$:

$$\left.\frac{d\mathcal{W}(r_0,t_0|r,r',t)}{dt}\right|_{t=t_0}\equiv\hat{\mathcal{L}}(r_0,t_0|r,r')\mathcal{W}(r_0,t_0|r,r',t_0). \tag{7}$$

The identification of the Liouville operator follows from the definitions (2) and (4),

$$\frac{dv(r_0,t_0|r-\rho(r_0,t),t],t)}{dt}\equiv\frac{du(r,t)}{dt}$$
$$=\frac{\partial u(r,t)}{\partial t}+[u(r,t)\cdot\nabla]u(r,t) \tag{8}$$

Translating all coordinates by $\rho(r_0,t)$ we find

$$\hat{\mathcal{L}}(r_0,t_0|r,r')=\left.\frac{\partial}{\partial t}\right|_{t=t_0}+v(r_0,t_0|r,t_0)\cdot\nabla_r$$
$$+v(r_0,t_0|r',t_0)\cdot\nabla_{r'}. \tag{9}$$

Consider now the time dependence of the ''fully fused'' second order correlation function

$$\mathcal{F}_2^{\alpha\beta}(r_0|r,r',\tau)=\langle\mathcal{W}^{\alpha}(r_0|r,r',t_0)\mathcal{W}^{\beta}(r_0|r,r',t_0+\tau)\rangle. \tag{10}$$

By ''fully fused'' we mean here that the space coordinates of the two velocity differences are the same, and they differ only in their time arguments. The same-time counterpart of this correlation function, i.e., $\mathcal{F}_2(r_0|r,r',\tau=0)$ is independent of r_0 and in an isotropic and homogeneous ensemble it is a function of $|r-r'|$ only. Accordingly it differs from the standard structure function $S_2(|r-r'|)$ only in having the full second rank tensorial character. For this analysis we choose all the three vector distances to have the modulus in the inertial range, of the order of R. For the sake of notation we will keep only this R but remember that the angular dependence is understood but not shown explicitly.

To proceed, we consider the tensorial correlation function (10) as an inner product in the state space of vectors $\mathcal{W}$, denoted as

$$\mathcal{F}_2(R,\tau)=(\mathcal{W},e^{\hat{\mathcal{L}}\tau}\mathcal{W}). \tag{11}$$

In particular we are interested in the Laplace transform of Eq. (11)

$$\tilde{\mathcal{F}}_2(R,z)=\left(\mathcal{W},\frac{1}{z-\hat{\mathcal{L}}}\mathcal{W}\right). \tag{12}$$

It has been shown by Grossmann and Thomae [6] that the Zwanzig-Mori projection operator formalism [8,9] applies to turbulent systems described by Navier-Stokes-like equations. In Ref. [7] it has been demonstrated that the contribution of the memory kernel and of the higher order continued fraction coefficients is considerable. The central idea is to decompose the resolvent

$$\hat{\mathcal{R}}(z)=\frac{1}{z-\hat{\mathcal{L}}} \tag{13}$$

by means of a projection operator $\hat{P}$ acting in state space. With $\hat{Q}=\hat{1}-\hat{P}$ as projector orthogonal to $\hat{P}$ one has the resolvent identity

$$\hat{P}\hat{\mathcal{R}}(z)\hat{P}=\hat{P}\frac{1}{z-\hat{P}\hat{\mathcal{L}}\hat{P}-\hat{P}\hat{\mathcal{L}}\hat{Q}[1/(z-\hat{Q}\hat{\mathcal{L}}\hat{Q})]\hat{Q}\hat{\mathcal{L}}\hat{P}}. \tag{14}$$

Since we note that the correlation function of interest -eq. (12)- is the $\mathcal{W}$-$\mathcal{W}$ matrix element of the resolvent we choose $\hat{P}$ to be the projector on $\mathcal{W}$,

$$\hat{P}\cdot\equiv\mathcal{W}(\mathcal{W},\mathcal{W})^{-1}(\mathcal{W},\cdot). \tag{15}$$

Its basic properties are $\hat{P}\hat{P}=\hat{P}$ (idempotent) and $\hat{P}^{\dagger}=\hat{P}$ (self-adjoint), characterizing an orthogonal projection. Now Eq. (14) yields the following expression for $\hat{P}$:

$$\hat{P}=\hat{P}\hat{\mathcal{R}}(z)\hat{P}\left[z-\hat{P}\hat{\mathcal{L}}\hat{P}-\hat{P}\hat{\mathcal{L}}\hat{Q}\frac{1}{z-\hat{Q}\hat{\mathcal{L}}\hat{Q}}\hat{Q}\hat{\mathcal{L}}\hat{P}\right]. \tag{16}$$

Computing the $\mathcal{W}$-$\mathcal{W}$ matrix element of the previous equation gives

$$(\mathcal{W},\mathcal{W})=[\mathcal{W},\hat{\mathcal{R}}(z)\mathcal{W}](\mathcal{W},\mathcal{W})^{-1}\Bigg[z(\mathcal{W},\mathcal{W})-(\mathcal{W},\hat{\mathcal{L}}\mathcal{W}) - \left(\mathcal{W},\hat{\mathcal{L}}\hat{Q}\frac{1}{z-\hat{Q}\hat{\mathcal{L}}\hat{Q}}\hat{Q}\hat{\mathcal{L}}\mathcal{W}\right)\Bigg]. \tag{17}$$

Hence one obtains

$$\tilde{\mathcal{F}}_2(R,z)=\frac{k_0(R)}{z-\gamma_0(R)-\tilde{K}_0(R,z)}, \tag{18}$$

where

$$k_0(R)=(\mathcal{W},\mathcal{W}),$$

$$\gamma_0(R)=(\mathcal{W},\hat{\mathcal{L}}\mathcal{W})/k_0,$$

$$\tilde{K}_0(R,z)=\left(\hat{Q}^{\dagger}\hat{\mathcal{L}}^{\dagger}\mathcal{W},\frac{1}{z-\hat{Q}\hat{\mathcal{L}}\hat{Q}}\hat{Q}\hat{\mathcal{L}}\mathcal{W}\right)/k_0. \tag{19}$$

Notice that $k_0,\gamma_0,\tilde{K}_0$ are tensors as $\tilde{\mathcal{F}}_2$ is itself so that Eq. (18) is a shorthand notation for

$$\tilde{\mathcal{F}}_2^{\alpha\beta}(R,z)=\frac{k_0^{\alpha\beta}(R)}{z-\gamma_0^{\alpha\beta}(R)-\tilde{K}_0^{\alpha\beta}(R,z)}. \tag{20}$$

Here, of course, $\hat{Q}^{\dagger}=\hat{Q}$, but in the next steps of generating the continued fraction hermiticity will not hold if $\hat{\mathcal{L}}$ and its adjoint $\hat{\mathcal{L}}^{\dagger}$ do not identify directly or up to a sign. Realizing that the kernel $\tilde{K}_0(R,z)$ has the same resolvent structure as $\tilde{\mathcal{F}}_2(R,z)$ except that it now features $\hat{Q}\hat{\mathcal{L}}\hat{Q}$ instead of $\hat{\mathcal{L}}$ and that the states are $\hat{Q}^{\dagger}\hat{\mathcal{L}}^{\dagger}\mathcal{W}$ as the bra vectors and $\hat{Q}\hat{\mathcal{L}}\mathcal{W}$ as the ket vectors instead of $\mathcal{W}$ and $\mathcal{W}$, one can continue the fraction by the same procedure. This is more transparent if we denote

$$\mathcal{W}_1=\hat{Q}\hat{\mathcal{L}}\mathcal{W},\quad \tilde{\mathcal{W}}_1=\hat{Q}^{\dagger}\hat{\mathcal{L}}^{\dagger}\mathcal{W},\quad \hat{\mathcal{L}}_1=\hat{Q}\hat{\mathcal{L}}\hat{Q} \tag{21}$$

so that $\tilde{K}_0(R,z)$ takes on the form

$$\tilde{K}_0(R,z)=\left(\tilde{\mathcal{W}}_1,\frac{1}{z-\hat{\mathcal{L}}_1}\mathcal{W}_1\right)/k_0. \tag{22}$$

Now we define a new projection operator

$$\hat{P}_1\cdot\equiv\mathcal{W}_1(\tilde{\mathcal{W}}_1,\mathcal{W}_1)^{-1}(\tilde{\mathcal{W}}_1,\cdot). \tag{23}$$

Because $\hat{\mathcal{W}}_1$ is different from $\mathcal{W}_1$ when $\hat{\mathcal{L}}$ is not Hermitian or anti-Hermitian, this operator $\hat{P}_1$ is not Hermitian and performs accordingly non-orthogonal projections. But $\hat{P}_1$ still is idempotent, $\hat{P}_1\hat{P}_1=\hat{P}_1$, which is the essential property for deriving the analogous resolvent identity with $\hat{\mathcal{L}}_1$ in Eq. (22) as for the original resolvent (12) with $\hat{\mathcal{L}}$. Defining $\hat{Q}_1\equiv 1-\hat{P}_1$ we can repeat the argument leading to Eqs. (18) and (19), and find

$$\tilde{K}_0(R,z)=\frac{k_1(R)}{z-\gamma_1(R)-\tilde{K}_1(R,z)}, \tag{24}$$

where

$$k_1(R)=(\tilde{\mathcal{W}}_1,\mathcal{W}_1)/k_0,$$

$$\gamma_1(R)=(\tilde{\mathcal{W}}_1,\hat{\mathcal{L}}_1\mathcal{W}_1)/k_1k_0,$$

$$\tilde{K}_1(R,z)=\left(\hat{Q}_1^{\dagger}\hat{\mathcal{L}}_1^{\dagger}\tilde{\mathcal{W}}_1,\frac{1}{z-\hat{Q}_1\hat{\mathcal{L}}_1\hat{Q}_1}\hat{Q}_1\hat{\mathcal{L}}_1\mathcal{W}_1\right)\Big/ k_1k_0. \tag{25}$$

Hence, repeating this procedure the Laplace transform $\tilde{\mathcal{F}}_2(R,z)$ of the correlation function (10) can be written in continued fraction representation:

$$\hat{\mathcal{F}}_2(R,z)=\cfrac{k_0(R)}{z-\gamma_0(R)-\cfrac{k_1(R)}{z-\gamma_1(R)-\cfrac{k_2(R)}{z-\gamma_2(R)-\ddots}}}. \tag{26}$$

In this expression we used, for $n=1,2,\ldots$, the notation

$$k_n(R)=(\tilde{\mathcal{W}}_n,\mathcal{W}_n)/k_{n-1}k_{n-2}\cdots k_0,$$

$$\gamma_n(R)=(\tilde{\mathcal{W}}_n,\hat{\mathcal{L}}_n,\mathcal{W}_n)/k_nk_{n-1}\cdots k_0, \tag{27}$$

where

$$\mathcal{W}_n=\hat{Q}_{n-1}\hat{\mathcal{L}}_{n-1}\mathcal{W}_{n-1},\quad \tilde{\mathcal{W}}_n=\hat{Q}_{n-1}^{\dagger}\hat{\mathcal{L}}_{n-1}^{\dagger}\tilde{\mathcal{W}}_{n-1},$$

$$\hat{\mathcal{L}}_n=\hat{Q}_{n-1}\hat{\mathcal{L}}_{n-1}\hat{Q}_{n-1}. \tag{28}$$

At each iteration of the procedure new projection operators are defined as

$$\hat{P}_n \cdot = \mathcal{W}_n(\tilde{\mathcal{W}}_n, \mathcal{W}_n)^{-1}(\tilde{\mathcal{W}}_n, \cdot)$$

$$\hat{Q}_n = \hat{1} - \hat{P}_n. \tag{29}$$

The novelty when the operator $\hat{\mathcal{L}}$ is not Hermitian or anti-Hermitian as is the case with the operator $\hat{\mathcal{L}}$ defined in Eq. (9) is that the new projection operators introduced to continue the fraction are not self-adjoint, performing thus nonorthogonal projections. Only the operators $\hat{P}$ and $\hat{Q}$ perform orthogonal projections. In the following section we analyze the scaling properties of this representation.

III. SCALING PROPERTIES OF THE CONTINUED FRACTION REPRESENTATION

In this section we determine the leading scaling exponents of the quantities $k_n(R)$ and $\gamma_n(R)$ which appear in the continued fraction expansion (26). Notice that these quantities are tensors as the correlation function Eq. (12) itself is. We shall show that the leading scaling behavior is given in terms of correlation functions of time derivatives of the velocity differences $\mathcal{W}$ computed at zero time differences:

$$k_n(R) \approx \langle \hat{\mathcal{L}}^n \mathcal{W}, \hat{\mathcal{L}}^n \mathcal{W} \rangle / k_{n-1} k_{n-2} \cdots k_0,$$

$$\gamma_n(R) \approx \langle \hat{\mathcal{L}}^n \mathcal{W} \hat{\mathcal{L}}^{n+1} \mathcal{W} \rangle / k_n k_{n-1} \cdots k_0. \tag{30}$$

Here the symbol $\approx$ means ''in leading scaling order'' in the limit $\mathrm{Re} \to \infty$ and we recall from Eq. (7) that

$$\hat{\mathcal{L}}(r_0, t_0 | r, r') \mathcal{W}(r_0, t_0 | r, r', t_0) = \left. \frac{d\mathcal{W}(r_0, t_0 | r, r', t)}{dt} \right|_{t=t_0}. \tag{31}$$

To demonstrate the correctness of the formulas (30) let us consider first the tensor k_0 defined in Eq. (19). Because of isotropy [see remark below Eq. (10)] its scaling is the same as that of the second order structure function S_2 given in Eq. (1)

$$k_0(R) = (\mathcal{W}, \mathcal{W}) \propto R^{\zeta_2}. \tag{32}$$

As for the tensor γ_0 notice that it features the operator $\hat{\mathcal{L}}$

$$\gamma_0(R) = (\mathcal{W}, \hat{\mathcal{L}} \mathcal{W}) / k_0. \tag{33}$$

We recall that from the scaling point of view in the limit of infinite Reynolds number the operator $\hat{\mathcal{L}}$ within the inner product (or equivalently within the averaging operation) simply amounts to multiply with $\mathcal{W}/R$, see Ref. [3]. This follows from the convergence in the *UV* and in the *IR* (in the limit $\mathrm{Re} \to \infty$) of the integral implied by the terms $\vec{\mathcal{P}} \mathcal{W} \cdot \nabla_r$ in Eq. (5), so that the leading contribution comes from distances of the order of R. In other words, the operator $\hat{\mathcal{L}}$ in a correlation function, when it operates on $\mathcal{W}$, introduces a term of the order of $\mathcal{W} \cdot \nabla \mathcal{W}$, which, due to the demonstrated locality in scale space, can be estimated as adding to the correlation a factor of the order of $\mathcal{W}/R$. Hence using Eq. (1) one obtains

$$\gamma_0(R) \propto R^{\zeta_3 - \zeta_2 - 1}. \tag{34}$$

We now turn to the tensor k_1 defined in Eq. (25) which using Eq. (21) is written as

$$k_1(R) = (\hat{Q}^\dagger \hat{\mathcal{L}}^\dagger \mathcal{W}, \hat{Q} \hat{\mathcal{L}} \mathcal{W}) / k_0 = (\mathcal{W}, \hat{\mathcal{L}} \hat{Q} \hat{\mathcal{L}} \mathcal{W}) / k_0. \tag{35}$$

The second equality results from the fact that $\hat{Q}$ is idempotent. Using the definition of $\hat{Q} = 1 - \hat{P}$ one gets two terms which contribute. Applying then the explicit form of the projector $\hat{P}$ one obtains for these two terms

$$k_1(R) = (\mathcal{W}, \hat{\mathcal{L}}^2 \mathcal{W}) / k_0 - (\mathcal{W}, \hat{\mathcal{L}} \mathcal{W})^2 / k_0^2. \tag{36}$$

Hence one obtains the following scaling:

$$k_1(R) \propto R^{\zeta_4 - \zeta_2 - 2}(1 + \mathrm{const}\, R^{\zeta_3 - \zeta_2 - (\zeta_4 - \zeta_3)}). \tag{37}$$

The leading term on the right-hand side of Eq. (37) can be determined using the inequality

$$\zeta_m - \zeta_{m-1} \leqslant \zeta_n - \zeta_{n-1}, \quad m > n. \tag{38}$$

This follows from the convexity of the structure function exponent ζ_n as a function of n, which in turn is an immediate consequence of Schwarz' inequality applied to any nth order structure function decomposition into $n = m_1 + m_2$ order ones together with choosing the outer integral scale L and not the inner scale η as the reference length.

$$|S_n(R)| = |\langle \mathcal{W}^{m_1 + m_2} \rangle| \leqslant \langle \mathcal{W}^{2m_1} \rangle^{1/2} \langle \mathcal{W}^{2m_2} \rangle^{1/2}$$

$$= (S_{2m_1} S_{2m_2})^{1/2}. \tag{39}$$

Scalingwise, $S(R) \propto (R/L)^\zeta$, again using L as the reference length. Because $R/L \leqslant 1$, Eq. (39) implies

$$\zeta_{m_1 + m_2} \geqslant \frac{\zeta_{2m_1} + \zeta_{2m_2}}{2}, \tag{40}$$

the defining property of convex functions. These enjoy a monotonously nonincreasing derivative, i.e., Eq. (38).

The contributions of the two terms in Eq. (37) can be written as $1 + \mathrm{const}(R/L)^x$, with $x = \zeta_3 - \zeta_2 - (\zeta_4 - \zeta_3)$. In the monofractal case x is zero, thus both terms scale alike. If there is multifractality, the nonlinear intermittency corrections make $x \neq 0$. It now depends on the sign of x and again on the choice of the reference length for the multifractal scaling, whether the second term is subdominant or dominant. If the reference length is η, the inner viscous scale, then $R/\eta > 1$ and the second term dominates for positive x. If, instead, the reference length is L, we have $R/L < 1$, implying that the second term is subdominant for positive x. But indeed $x \geqslant 0$ is a consequence of Schwarz' inequality which together with identifying L as the reference length implies Eq. (38). That L is the relevant reference length is under discussion but there are plausible arguments [14] though no rigorous proof. Positive x together with L as the proper reference length for intermittency then also imply that the second term in Eq. (37) can be neglected if R is well within the inertial range.

One obtains then

$$k_1(R) \approx (\mathcal{W}, \hat{\mathcal{L}}^2 \mathcal{W}) / k_0 \propto R^{\zeta_4 - \zeta_2 - 2}. \tag{41}$$

We are now in the position to determine the scaling of all the tensors k_n and γ_n appearing in Eq. (26). Proceeding along the same lines one gets

$$k_n(R)\approx(\mathcal{W},\hat{\mathcal{L}}^{2n}\mathcal{W})/k_{n-1}k_{n-2}\cdots k_0,$$
$$\gamma_n(R)\approx(\mathcal{W},\hat{\mathcal{L}}^{2n+1}\mathcal{W})/k_nk_{n-1}\cdots k_0. \tag{42}$$

This result can also be obtained by induction [13]. Eq. (42) together with Eq. (1) yield then the following explicit scaling for $n=1,2,\ldots$:

$$k_n(R)\propto R^{\zeta_{2n+2}-\zeta_{2n}-2},$$
$$\gamma_n(R)\propto R^{\zeta_{2n+3}-\zeta_{2n+2}-1}. \tag{43}$$

In finishing this section we note that the essence of the argument is that for all n the projectors $\hat{Q}_n=1-\hat{P}_n$ can scalingwise be approximated as 1, since $\hat{P}_n\propto(R/L)^x$ and $x=\zeta_{j+1}-\zeta_j-(\zeta_{j+2}-\zeta_{j+1})$ for some appropriate j. The terms omitted in comparison to the leading scaling order ones are negligible only if there is substantial multifractality. They may contribute at the inner scale border of the inertial range, i.e., if R approaches η from above. The larger Re, the longer is the inertial subrange, the better is the leading order approximation.

In case that η would be the relevant reference length the leading scaling approximation of the continued fraction coefficients would be different, resulting in a different multiscaling of the correlations. These differences can be subjected to experimental test.

In closing this section we emphasize that in the monofractal case we have to expect that all terms resulting from the decomposition of the Q projectors contribute alike, as was observed already in Ref. [7]. Since the intermittency corrections, which are responsible for x being nonzero, are small, for the physically realizable Reynolds numbers Re one still will need all terms. It is only asymptotically for large enough Re that the leading scaling order will suffice. It is the advantage of the continued fraction expansion as compared to the multifractal representation that all corrections are fully included.

IV. SCALING LAWS IMPLIED BY THE CONTINUED FRACTION REPRESENTATION: DERIVATIVES AT TIME ZERO

In this section we identify the leading scaling exponents that characterize the nth order time derivative of the correlation function Eq. (11) at $\tau=0$. We show that in the limit $\mathrm{Re}\to\infty$

$$\left.\frac{\partial^n\mathcal{F}_2(R,\tau)}{\partial\tau^n}\right|_{\tau=0}\propto R^{\zeta_{2+n}-n}n=0,1,\ldots. \tag{44}$$

From Eq. (18) one deduces by inverse Laplace transform the following equation:

$$\frac{\partial\mathcal{F}_2(R,\tau)}{\partial\tau}=\gamma_0(R)\mathcal{F}_2(R,\tau)+\int_0^\tau K_0(R,\tau')\mathcal{F}_2(R,\tau-\tau')d\tau', \tag{45}$$

where $\mathcal{K}_0(R,\tau)$ is the inverse Laplace transform of $\tilde{\mathcal{K}}_0(R,z)$, see Eq. (22), i.e.,

$$K_0(R,\tau)=\frac{1}{k_0(R)}(\tilde{\mathcal{W}}_1,e^{\hat{\mathcal{L}}_1\tau}\mathcal{W}_1). \tag{46}$$

Equation (45) is the so-called memory-function equation, $\mathcal{K}_0(R,\tau)$ being the memory kernel. At $\tau=0$ Eq. (45) becomes

$$\frac{\partial\mathcal{F}_2}{\partial\tau}(R,0)=\gamma_0(R)k_0(R)\propto R^{\zeta_3-1}, \tag{47}$$

where we used the scaling laws (32), (34).

The second order partial time derivative is obtained by differentiating Eq. (45),

$$\frac{\partial^2\mathcal{F}_2(R,\tau)}{\partial\tau^2}=\gamma_0(R)\frac{\partial\mathcal{F}_2}{\partial\tau}(R,\tau)+K_0(R,\tau)\mathcal{F}_2(R,0)+\int_0^\tau d\tau' K_0(R,\tau')\frac{\partial\mathcal{F}_2}{\partial\tau}(R,\tau-\tau'). \tag{48}$$

At $\tau=0$ from Eqs. (25), (46), (47) one obtains

$$\frac{\partial^2\mathcal{F}_2}{\partial\tau^2}(R,0)=k_0(R)[\gamma_0^2(R)+k_1(R)], \tag{49}$$

which, using Schwarz' inequality in the form (38) together with $R/L\leqslant1$ in the inertial subrange, is found to scale in leading order as

$$\frac{\partial^2\mathcal{F}_2}{\partial\tau^2}(R,0)\approx k_0(R)k_1(R)\propto R^{\zeta_4-2}. \tag{50}$$

Differentiating Eq. (48) once more and evaluating at $\tau=0$ one finds

$$\frac{\partial^3\mathcal{F}_2}{\partial\tau^3}(R,0)\approx k_0(R)k_1(R)\gamma_1(R)\propto R^{\zeta_5-3}. \tag{51}$$

One arrives thus at

$$\frac{\partial^{2n}\mathcal{F}_2}{\partial\tau^{2n}}(R,0)\approx k_0(R)\cdots k_n(R)\propto R^{\zeta_{2+2n}-2n}$$
$$\frac{\partial^{2n+1}\mathcal{F}_2}{\partial\tau^{2n+1}}(R,0)\approx k_0(R)\cdots k_n(R)\gamma_n(R)\propto R^{\zeta_{3+2n}-2n-1}, \tag{52}$$

which can also be proven more formally by complete induction [13]. The general expression comprising these $n=1,2,3$ cases can be written in the form of Eq. (44).

We have thus derived the complete Taylor series of the dynamical correlation function $\mathcal{F}_2(R,\tau)$, in which not only the scaling behavior of the coefficients but also their absolute magnitude can be evaluated,

$$\mathcal{F}_2(R,\tau)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n\mathcal{F}_2}{\partial\tau^n}(R,0)\tau^n=\sum_{n=0}^{\infty}\frac{1}{n!}A_n(R/L)^{\zeta_{n+2}-n}\tau^n. \tag{53}$$

In the special case of monofractality in the spatial scaling we obtain also temporal monoscaling from Eq. (53). Let $\zeta_n = n\zeta_1$, then $\zeta_{2+n} - n = 2\zeta_1 - n(1-\zeta_1)$. Therefore, the monofractal form arises

$$\mathcal{F}_2(R,\tau) = R^{2\zeta_1} f_2(\tau/R^{1-\zeta_1}), \tag{54}$$

where f_2 is a function of the scaled time variable only. The dynamical scaling exponent in the monofractal case is thus $z = 1-\zeta_1$ while the static exponents are $\zeta_n = n\zeta_1$.

V. THE MULTISCALING REPRESENTATION

In this section we compare two independent expressions for the correlation function $\mathcal{F}_2(R,\tau)$: on the one hand the multiscaling representation considered in Ref. [3] (which has not been used here so far despite the fact that we used the word multiscaling in the text above) and on the other hand the continued fraction representation derived here. The multiscaling representation of $\mathcal{F}_2(R,\tau)$ can be written as [3]

$$\mathcal{F}_2(R,\tau) = U^2 \int d\mu(h) \left(\frac{R}{L}\right)^{2h+\mathcal{Z}(h)} f_2\left(\frac{\tau}{\tau_{R,h}}\right), \tag{55}$$

where U is the characteristic magnitude of the velocity difference across the outer scale of turbulence, f_2 is a function of the scaled time variable only, and

$$\tau_{R,h} \sim \frac{R}{U}\left(\frac{L}{R}\right)^h. \tag{56}$$

The function $\mathcal{Z}(h)$ is related to the scaling exponents ζ_n of the nth order structure functions through the saddle point requirement

$$\zeta_n = \min_h [nh + \mathcal{Z}(h)]. \tag{57}$$

To find the scaling exponents associated with the time derivatives of $\mathcal{F}_2(R,\tau)$ at $\tau=0$ one computes the nth order partial time derivative of Eq. (55) to obtain

$$\frac{\partial^n \mathcal{F}_2(R,\tau)}{\partial \tau^n} = \frac{U^{2+n}}{R^n} \int d\mu(h) \left(\frac{R}{L}\right)^{(2+n)h+\mathcal{Z}(h)} \frac{\partial^n f_2}{\partial \tau^n}\left(\frac{\tau}{\tau_{R,h}}\right). \tag{58}$$

At $\tau=0$ this gives

$$\frac{\partial^n \mathcal{F}_2}{\partial \tau^n}(R,0) = \frac{\partial^n f_2}{\partial \tau^n}(0) \frac{U^{2+n}}{R^n} \int d\mu(h) \left(\frac{R}{L}\right)^{(2+n)h+\mathcal{Z}(h)}. \tag{59}$$

Computing the integral at the saddle point in the limit $R/L \to 0$ and using Eq. (57) we find

$$\frac{\partial^n \mathcal{F}_2}{\partial \tau^n}(R,0) \propto R^{\zeta_{2+n}-n}. \tag{60}$$

We notice that this scaling is the same as in Eq. (44) obtained from the continued fraction representation of $\mathcal{F}_2$ which is derived independently of the multiscaling representation. We thus see that the leading scaling approximation of the continued fraction expansion generates the same predictions regarding the multiplicity of time scales characterizing the time correlation functions as the multiscaling representation. We take this as an independent evidence for the correctness of the latter. If corrections to scaling become relevant, the continued fraction form, being exact, has to be taken.

VI. CONCLUSION

In conclusion, we showed that the exact continued fraction expansion of the time correlation functions of the Monin-Belinicher-L'vov-velocity differences has the same Taylor expansion as the multiscaling representation of these correlation functions, order by order in terms of the leading scaling contributions. We wish to emphasize that the continued fraction representation can be used as an approximant for the time correlation function, when analytic forms of such time-correlation functions are needed. In the lowest order continued fraction approximation one takes in Eq. (26) $k_1(R)=0$, producing an exponential temporal decay of the correlation function, with the decay rate $\gamma_0(R) \propto R^{\zeta_3-\zeta_2-1}$. The next approximation, $k_2=0$, is written as

$$\tilde{\mathcal{F}}_2(R,z) = \frac{k_0(R)}{z-\gamma_0(R)-k_1(R)/[z-\gamma_1(R)]}. \tag{61}$$

In every successive approximation ($k_2=0$, $k_3=0$, etc.) one introduces more and more characteristic scales, each one characterized by a different ''dynamical exponent,'' taking progressively more information about the statistics of higher order correlation functions into account.

Each finite, mth order continued fraction approximation results in an m-pole representation, whose poles as well as residues can be calculated from the static moments up to the order $2m$. The temporal correlation decay is a superposition of m exponentials with calculable decay rates.

As a final remark we stress that the conclusion is valid forward and backward: nonlinear or multiscaling ζ_n of the spatial scaling of the correlation functions imply temporal multiscaling and vice versa, i.e., temporal monoscaling (with some scaling exponent z) is consistent only with linear, monoscaling $\zeta_n = n\zeta_1$ in spatial scaling. This sufficient and necessary relation follows from the uniqueness of the continued fraction expansion. Namely, it is the very essence of the continued fraction representation that it completely determines all dynamical features from the static, i.e., equal time correlators.

One can easily identify the temporal scaling exponent z defined in Eq. (1) or its negative $a=-z$ for the λ-rescaling exponent of the Laplace space frequency, if the static moments obey monoscaling. Start by convincing yourself that all γ_n scale as $\hat{\mathcal{L}}_n$, which behave as $\hat{\mathcal{L}}$ itself, because scalingwise the Q's are 1, thus all $\gamma_n \propto \mathcal{W}/R \propto R^{\zeta_1-1}$. Next verify that all k_n scale as $\hat{\mathcal{L}}\hat{\mathcal{L}}$. When rescaling $\mathcal{F}_2(\lambda R, \lambda^a z)$, one has to compare $\lambda^a z$ with $\lambda^{\zeta_1-1}\gamma$ order by order of the continued fraction, implying $a=\zeta_1-1$. (The numerator in each memory kernel k_n is $\propto \hat{\mathcal{L}}\hat{\mathcal{L}}$, the corresponding denominator $\propto \hat{\mathcal{L}}$, thus each nth order memory contribution behaves like $\hat{\mathcal{L}}$, as the γ's do.)

If there are intermittency corrections, meaning nonlinear n dependence of the ζ_n, the γ_n are $\propto \hat{\mathcal{L}}_n$ and the $k_n \propto \hat{\mathcal{L}}_n \hat{\mathcal{L}}_n$, being nth or $2n$th order moments, all scale differently with R. Each continued fraction order thus has different R-scaling coefficients, therefore breaking any clearcut scaling of the Laplace variable z. Note that the ''amount'' of breaking the z or τ scaling is of the order of the range in which the ζ_n differ.

It should be clear that the lowest order continued fraction, putting $\tilde{\mathcal{K}}_0 = 0$, still enjoys time scaling, because only one moment, $\gamma_0(R)$, determines the dynamics. It also means that any higher order truncation of the continued fraction influences the value of the approximate τ exponent. One needs the full continued fraction Eq. (26) to obtain the correct τ behavior.

In closing we first point out that besides measuring higher order static equal time moments to determine the continued fraction coefficients experimentalists should feel highly encouraged to measure second order time correlation functions [cf. Eq. (10)] as functions of both the spatial scale R and the time τ, because on the latter it has a nontrivial functional dependence reflecting spatial multifractality in the dynamics. Secondly, the extension of the continued fraction expansion to non-Hermitian dynamics, given here for the first time to the best of our knowledge, should prove useful for other physical systems sharing this property.

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